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Multidiffusion in critical dynamics of strings and membranes

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We study dynamical roughening of strings which do not break the symmetry either parallel or vertical to the overall suspension. The suggested nonlocal dynamics enforces a buildup of long-range correlations and the system achieves self-organized criticality with a universality class which exhibits nontrivial temporal scalings and power-law-correlated activity along the string, even though only localized Gaussian noise is applied. We observe multiscaling of the temporal behavior both perpendicular and parallel (multidiffusion) to the evolving string at criticality.

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Dynamical roughening of fronts and interfaces has been studied quite intensively in recent years by using stochastic [1, 2], deterministic [3], as well as various deposition models [4, 5]. In all of these models one considers the temporal progression of a front $h(x,t)$, $x \in [1, L]$, where $\langle h \rangle$ increases with time t. Thus the $h \to -h$ symmetry is broken, implying that the large-scale evolution may be governed by the Karder-Parisi-Zhang (KPZ) equation: $dh/dt = \Delta h + (dh/dx)^2 + \eta(x,t)$. In contrast to a progressing front, we consider in this paper the dynamical evolution of a string $\{h(x)\}$ constrained to a plane with a unidirectional geometry (no overhangs). The important difference to standard interface models is that the updating rules are not only symmetric under $x \to -x$ but also under $h \to -h$. In the simplest case where subsequent wrinkles on the string are completely uncorrelated, the large scale evolution of the string $\{x, h(x)\}$ can be described by the Edwardson-Wilkinson (EW) equation $dh/dt = \Delta h + \eta(x,t)$; see Ref. [6]. The concept may easily be generalized to the roughening dynamics of surfaces or membranes in higher dimensions.

In contrast to the simplest possible symmetric dynamics, where the motion of the string appears randomly uncorrelated, we consider here the case where wrinkles on the string always happen at the point of global minimum η along the string. We demonstrate in this paper that this specific and conceptually simple nonlocal dynamics leads to a previously unreported class of critical exponents. The model is a symmetric analog to the asymmetric interface model of Ref. [7], where the activity of the interface appeared exactly at the point where the pinning was minimal [7, 8].

The model is defined on a lattice with sites (x, h). In the one-dimensional version a discrete string $h(x)$ is defined on $x = 1, 2, 3, ..., L$. Along the string a sequence of uncorrelated Gaussian random numbers η is distributed. We use periodic boundary conditions. The chain is updated, using a global comparison, by finding the site with the smallest random number η among all sites on the string. On this site one chooses with equal probability to either add or subtract 1 from h. Next, the neighboring sites are adjusted in the same direction as the chosen site, precisely until all slopes $|h(x) - h(x-1)| \leq 1$. This creates a local burst of activity that in simulations appears exponentially bounded. New random η's are assigned to all newly adjusted sites. Thus the updating algorithm is symmetric in h.

Notice the difference to the asymmetrical model of Ref. [7] which does not depend on whether the noise is generated along the updating dynamics or already was predetermined at the beginning. In contrast, the symmetric model is defined through the subsequent generation of the noise. Otherwise the activity would get trapped by the quenched noise in localized regions around the smallest η values, and the scalings would become widely different from what we observe in the present model. One may interpret the η as barriers for a dynamics driven by fluctuations much smaller than η, like proposed in the evolution dynamics of Ref. [9]. Thus the present model can be understood as describing the dynamics of random barriers which can modify each other according to the h field.

It is important to stress that the crucial ingredient of the dynamics is the *global* comparison of the barriers η. Numerically we find that without the global comparison of η, then independently on whether the noise is quenched or generated along the updated string, the large scale behavior appears to be identical to the one of the EW equation.

Consider now the large scale evolution of a string governed by the symmetric model defined above. Initially, the string roughens in a transient towards the saturated state. We restrict to the saturated states in the following. First notice from simulations of saturated systems with $L = 32$ to $L = 1024$ that the width $w = \langle (h - \langle h \rangle)^2 \rangle^{1/2} \propto L^{\chi}$ with $\chi = 0.46 \pm 0.04$ whereas the infinite moment $H = \langle \max(h) - \min(h) \rangle \propto L^{0.50 \pm 0.03}$. Thus there is no significant indication of spatial multiscaling [10] and the scaling exponent is compatible with the one predicted by the EW equation where $\chi = \frac{1}{2}$. The transient time is denoted τ and moments of the height-height time correlations [11, 12] are measured by ensemble averaging over saturated states

$$W_q(L,t) = \langle ([h(x,t+\tau) - h(x,\tau)] - \langle h(x,t+\tau) - h(x,\tau) \rangle)^q \rangle^{1/q} . \quad (1)$$

From Fig. 1 we observe for the second moment

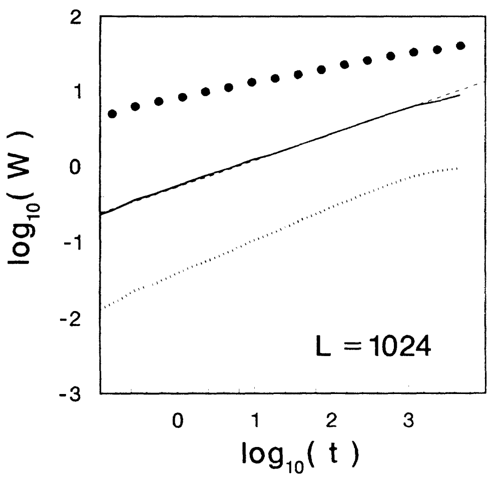

FIG. 1. Scaling of height-height correlations with time (for system size $L = 1024$), started at saturated time: The full line shows the second moment W, the solid circles the ∞ moment $H(t)$, and the dotted line displays the ratio of sites where height has changed. To guide the eye is thin dashed lines with slope 0.35.

$W_2(L,t) \propto t^{\beta_2}$ with $\beta_2 = 0.35 \pm 0.02$. The infinite moment

$$H(t) = \langle\, \max_x\{h(x,t+\tau) - h(x,\tau)\} - \min_x\{h(x,t+\tau) - h(x,\tau)\}\rangle_\tau \qquad (2)$$

scales as $H(t) \propto t^{\beta_\infty}$ with $\beta_\infty = 0.20 \pm 0.03$. Also the zero moment, defined as the number of sites where changes have occurred, displays a clear scaling: $N(t) \propto t^{\beta_0}$ with $\beta_0 = 0.43 \pm 0.03$. This temporal multiscaling is beyond anything that could be described within the framework of Langevin equations with uncorrelated white noise.

In fact we will now, as in Ref. [12], consider the behavior of the activity along the string. In the transient, the activity along the string appears uncorrelated in space, but as time progresses subsequent activities get correlated over larger and larger distances. Finally, at saturation, a critical state is built up, in the sense that the subsequent spatial activity displayed in Fig. 2(a) exhibits scale invariance:

$$P(X) \propto X^{-3.1\pm0.2}. \qquad (3)$$

Because nontrivial spatial correlations naturally appear from the dynamics, the model exhibits self-organized critical behavior. These spatial correlations naturally induce some temporal correlations. For a given point x we monitor two quantities in the saturated critical state: The waiting times for subsequent activity and the probability for return since a certain activity. As a function of time t we observe for the corresponding distributions

$$P_W(t) \propto t^{-1.6\pm0.1} \quad \text{and} \quad P_A(t) \propto t^{-\tau_A}. \qquad (4)$$

The scaling exponent of the return activity $\tau_A = 0.42 \pm 0.02$ resembles the scaling of activated sites, i.e., $\tau_A \approx \beta_0$, which reflects that the temporal spreading of activity approximately balance the decrease in the local average activity. Furthermore notice that the exponent $\tau_A = 0.42 \pm 0.02$ is incompatible with uncorrelated subsequent jumps determined by the scaling of Eq. (3). In fact, as seen in Ref. [13], a Levy flight with spatial exponent $f = 2$ [corresponding to $P(X) \propto X^{-3}$] has a first passage probability with temporal exponent $\tau_F(\text{Levy}) = \min\{2 - 1/f, 1.5\} = 1.5$ and passage probability at all with exponent $\tau_A(\text{Levy}) = \max\{1/f, 0.5\} = 0.5$. Thus the activity observed in the symmetric model stays for longer time localized than expected from its spatial correlations alone.

To characterize the saturated state of the present model one may also investigate the distribution of η along the string. In Fig. 3 it is seen that the minimal η along the saturated strings does not exceed a certain threshold value, $\eta_{\text{crit}} \approx 0.74$, which reflects that all $\eta > \eta_{\text{crit}}$ effectively are "pinned." Notice that the value of η_{crit} is much higher than in the broken symmetry model of Ref. [7] where an analogy to directed percolation is used in Ref. [14] to prove that $\eta_{crit}(b) \approx 0.46$.

Another quantity of interest is the moments of accumulated activity along the string:

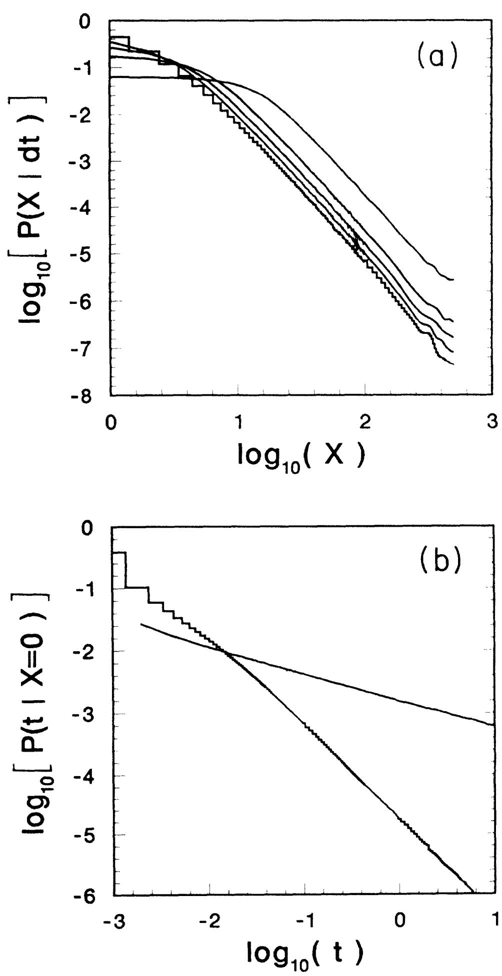

FIG. 2. (a) Spatial distribution of activity centers, separated in time by $dt = 1/L$, $2/L$, $4/L$, $10/L$, and $100/L$ with a system size $L = 1024$. (b) Probability for activity in a given site as function of time: "Steep" full line: first return; "Flat" full line: all returns.

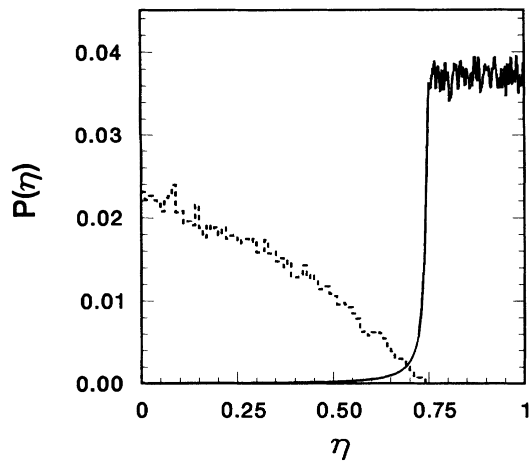

FIG. 3. Ensemble averaged distribution of noise η on a saturated interface (with length $L = 512$). Full line display histogram of all η, dashed only of minimal η. As input we used $\eta \in [0,1]$ homogeneously distributed. Notice the self-organized threshold for pinned sites $\eta_{\rm crit} = 0.74$ corresponding to a fraction of 0.26 inactive sites.

$$A_q(t) = \left\langle \left(\sum_{t'=\tau}^{\tau+t} |h(t'+1,x) - h(t',x)| \right)^q \right\rangle^{1/q} \propto t^{\zeta_q}. \tag{5}$$

As shown in Fig. 4 this quantity scales for all q over five orders of magnitude, with exponents $\zeta_1 = 1$ (per definition), $\zeta_2 = 0.80 \pm 0.01$, $\zeta_4 = 0.70 \pm 0.02$, and $\zeta_\infty = 0.61 \pm 0.02$ (and in fact $\zeta_0 = \beta_0 = 0.43 \pm 0.03$). The reason for being interested in the A_q's is that they directly associate with quantities dependent on the activity pattern, and that the A_q scaling differs in a nontrivial way from the height-height correlations. In contrast, for a progressing interface like in Ref. [7], the A_q and W_q would scale identically. An example where the pattern of accumulated activity is useful can be found in Ref. [9] which studies the evolution of a large number of systems, each with many metastable states, and organized such that the systems locally can modify each others barriers. For sufficiently low temperatures this collection of weakly coupled systems is governed by fluctuations connected to the passing of the overall lowest barrier, like in the present model. The accumulated quantity above is then associated to the counting of the number of times some of the systems have penetrated new barriers. The scaling exponents of the evolution model of Ref. [9] furthermore appear identical to exponents of the present string model and the two models therefore belong to the same new universality class.

We now investigate the scalings along the string. Whereas the previously measured exponents χ, β, and ζ relate to scalings perpendicular to the main direction of the suspended string, one may also study the scaling of displacement of a particle diffusing along the roughening string [15]. To do this we consider an ensemble of passively noninteracting particles that moves along saturated strings. For the motion of each of these particles alone we break the up down symmetry ($h \to -h$) by a dynamics that moves a particle at point x at each time step $[t, t+dt]$ a step $\Delta = h(x+1) - h(x-1)$. In Fig. 5 we display the average of the moments of the absolute displacement X of the particles as function af time t. We observe a first moment scaling $\langle X \rangle \propto t^{0.74\pm0.03}$, second moment scaling $\langle X^2 \rangle^{1/2} \propto t^{0.52\pm0.03}$, and an infinite moment scaling $\langle X_{\rm max} \rangle \propto t^{0.38\pm0.03}$. This multiscaling of the dynamical exponents Z_q defined by the scaling $\langle X^q \rangle^{1/q} \propto t^{1/Z_q}$ we call multidiffusion. This phenomena of widely different dynamical behaviors of typical and maximal displacements may have important implications for processes that depend, respectively, on a single particle or on the bulk properties of the passively "diffusing" particles.

It is emphasized that in contrast to what is expected

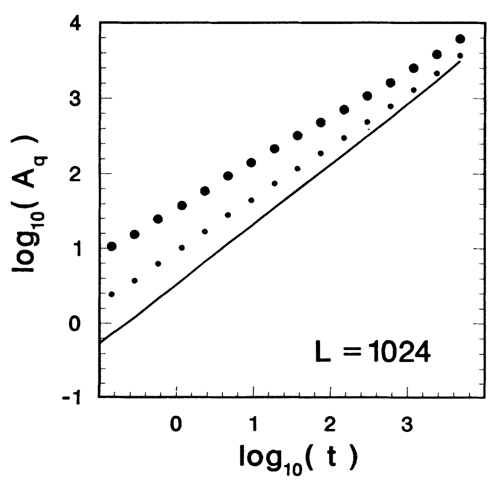

FIG. 4. Moments of accumulated absolute values of height changes. Full line for second moment, small black circles for fourth moment, and big black solid circles for infinite moment. System size $L = 1024$.

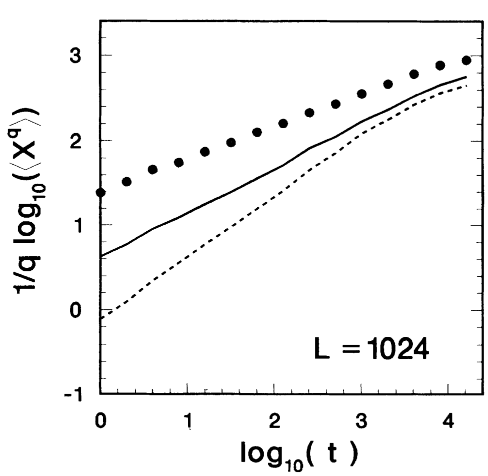

FIG. 5. Moments of total displacement of passively diffusing particles along the string. Dashed line is first moment, full line is second moment, and big black dots display the infinite moment.

for KPZ or EW like dynamics, we find $Z_2 \neq \chi/\beta_2$. Thus the relations between motions perpendicular to the string ($h \leftrightarrow t$ and $h \leftrightarrow x$) do not relate directly to the scaling of motion along the string.

To summarize, we have introduced a universality class in nonequilibrium statistical physics which, by using a simple nonlocal rule, develops critical states with self-organized thresholds, nontrivial scalings, and power-law-correlated activity. The model is studied for a one-dimensional string geometry but is easily generalized to higher dimensions, and may then define a new class of nontrivial roughening phenomena in surfaces and membranes. Furthermore, the nonequilibrium dynamics that drive the string or membrane to its self-organized critical state is symmetric in both space x and field h in contrast to the other models that dynamically develop criticality (see Refs. [16–18, 7]). Finally we introduced a new concept, multidiffusion, that describes processes where the extreme displacement of randomly moving particles scales differently than the typical displacement. We are in the progress of studying the model at finite fluctuations, and believe that the proposed nonlocal dynamics can appear from a local dynamics where the probability for spontaneous update of a site is determined by the passage probability over random barriers at very low temperatures.

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